# RECITATION 4

## THE DERIVATIVE

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## *Section* 1. *The Derivative*

The derivative is the slope of the tangent line on a graph. Intuitively, this is the somewhat dubious concept of an "instantaneous rate of change". More appropriately, it's a limit. In particular, it's a limit of approximating rates of change or slopes. Recall that the slope between two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is given by

$$\text{slope} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

The derivative at $x_1$ is then just the limit of the above as $x_2$ approaches $x_1$.

**1 • 1. Definition**

Let $f$ be a function. The *derivative* of $f$ at a point $x$, $f'(x)$, is the limit

$$\lim_{x_2 \to x} \frac{f(x) - f(x_2)}{x - x_2} = \lim_{\Delta x \to 0} \frac{f(x) - f(x + \Delta x)}{x - (x + \Delta x)} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

The function $f'$ is called the derivative of $f$, and the operation taking $f$ to $f'$ is called *differentiation*. For example, one differentiates $f$ to find the derivative $f'$.

We will learn rules to evaluates these limits far more easily, but we should always be able to resort to this definition, and understand how the limit works out.

Note that almost no functions are differentiable. Having a derivative is a very strong property that only sufficiently smooth functions have. Indeed, all differentiable functions are continuous, for example, although almost no functions are continuous[i]. In fact, most functions are best described as hazes of points without any connection between the points or formula defining them. Having things line up in a continuous way is itself very strong, and having a derivative is even stronger, basically saying that the function in question is "locally linear" in a precise way.

Note also that the definition of $f'$ is a limit, and so is really two one-sided limits in disguise. But this has the added benefit of not really adding anything new: we can calculate $f'(x)$ just the same as any other limit. This is especially true for complicated functions like in Exercise 7.

[i] One can show there are as many continuous functions as real numbers, but there are strictly more functions from reals to reals. In fact, consistently, this number can be larger than any particular cardinal number.

# *Section 2. Exercises*

**Exercise 1**

Is $f(x) = 1$ continuous at $x = 1$? Is $g(x) = \frac{x-1}{x-1}$?

*Solution* .:.

Yes 1 is continuous everywhere, since $\lim_{x\to a} 1 = 1 = f(a)$ for all $a$.
No, $\frac{x-1}{x-1}$ isn't even defined at $x = 1$. Despite the fact that $\lim_{x\to 1} \frac{x-1}{x-1} = 1$, $1 \neq g(1)$, since $g(1)$ doesn't exist.

**Exercise 2**

For which $A$ and $B$ is the function $f$ continuous? Here

$$f(x) = \begin{cases} Ax & \text{if } x < 0 \\ Bx & \text{if } x \geq 0. \end{cases}$$

*Solution* .:.

Since $Ax$ and $Bx$ are both continuous for all $A, B$, it suffices to ensure $A \cdot 0 = B \cdot 0$. But this is always true: it's equivalent to $0 = 0$. Hence $f$ is always continuous, no matter what $A$ and $B$ are.

**Exercise 3**

Consider the functions

$$f(x) = \begin{cases} 2x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \qquad g(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ -1 & \text{if } x = 0. \end{cases}$$

What are the discontinuities of $f$? of $g$? of $f + g$?

*Solution* .:.

$f$ is discontinuous at $x = 0$, since $\lim_{x\to 0} f(x) = \lim_{x\to 0} 2x = 0 \neq f(0) = 1$. The same idea tells us that $g$ is discontinuous at $x = 0$. $f + g$, however, is continuous everywhere, since

$$(f+g)(x) = f(x) + g(x) = \begin{cases} 2x + x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and $\lim_{x\to 0}(f+g)(x) = 2 \cdot 0 + 0^2 = 0 = (f+g)(0)$.

**Exercise 4**

Differentiate $f(x) = 5x$.

*Solution* .:.

Using a difference quotient,

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{5x + 5\Delta x - 5x}{\Delta x} = \frac{5\Delta x}{\Delta x}.$$

Hence if we take the limit as $\Delta x \to 0$,

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{5\Delta x}{\Delta x} = 5.$$

**Exercise 5**

Using the definition of the derivative, calculate $f'(1/4)$ where $f(x) = \sqrt{x}$.

*Solution* .:.

Using a difference quotient, noting that $1/4 + \Delta x \geq 0$ for sufficiently small $\Delta x$, allowing us to say

$\sqrt{1/4+\Delta x}^2 = |1/4+\Delta x| = 1/4+\Delta x$:

$$\frac{f\left(\frac{1}{4}+\Delta x\right)-f\left(\frac{1}{4}\right)}{\Delta x} = \frac{\sqrt{\frac{1}{4}+\Delta x}-\sqrt{\frac{1}{4}}}{\Delta x} = \frac{\sqrt{\frac{1}{4}+\Delta x}-\frac{1}{2}}{\Delta x}$$
$$= \frac{\frac{1}{4}+\Delta x-\frac{1}{4}}{\Delta x\cdot\left(\sqrt{\frac{1}{4}+\Delta x}+\frac{1}{2}\right)} = \frac{\Delta x}{\Delta x\cdot\left(\sqrt{\frac{1}{4}+\Delta x}+\frac{1}{2}\right)}.$$

So taking a limit yields

$$f'\left(\frac{1}{4}\right) = \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x\cdot\left(\sqrt{\frac{1}{4}+\Delta x}+\frac{1}{2}\right)} = \frac{1}{\sqrt{\frac{1}{4}}+\frac{1}{2}} = \frac{1}{2/2} = 1.$$

**Exercise 6**

Differentiate $f(x) = \frac{1}{x}$.

*Solution* .:.

Using a difference quotient,

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} = \frac{\frac{x-(x+\Delta x)}{(x+\Delta x)x}}{\Delta x} = \frac{-\Delta x}{\Delta x(x+\Delta x)x}$$

So taking a limit yields

$$f'(x) = \lim_{\Delta x\to 0}\frac{-\Delta x}{\Delta x(x+\Delta x)x} = \frac{-1}{x^2}.$$

**Exercise 7**

Consider $f(x) = \begin{cases} x & \text{if } x<0 \\ x^2+x & \text{if } x\geq 0\end{cases}$. Does $f'(0)$ exist?

*Solution* .:.

To determine whether $f'(0)$ exists, we need to make sure that the two one-sided limits exist:

$$\lim_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x} \quad\text{and}\quad \lim_{\Delta x\to 0^-}\frac{f(\Delta x)-f(0)}{\Delta x}.$$

Note that as $\Delta x$ approaches 0 from the right, $\Delta x > 0$. Hence we have

$$\lim_{\Delta x\to 0^+}\frac{f(\Delta x)-f(0)}{\Delta x} = \lim_{\Delta x\to 0^+}\frac{(\Delta x)^2+(\Delta x)-0}{\Delta x} = \lim_{\Delta x\to 0^+}\Delta x+1 = 1.$$

As $\Delta x$ approaches 0 from the left, $\Delta x < 0$, and hence we have

$$\lim_{\Delta x\to 0^-}\frac{f(\Delta x)-f(0)}{\Delta x} = \lim_{\Delta x\to 0^-}\frac{\Delta x-0}{\Delta x} = 1.$$

Hence the two limits are equal, and thus $f'(0)$ exists with value $f'(0) = \lim_{x\to 0}\frac{f(\Delta x)-f(0)}{\Delta x} = 1$.